4-10.1 Other Chebyshev Array Techniques [31, 32, p. 101]

We use the Chebyshev polynomials to arrive at other equal-sidelobe designs. The method of the preceding sections fails to give optimum designs for spacings less than λ/2. We need to restrict the designs to an odd number of elements to use the following methods. We start with the array factor an odd number of elements fed symmetrically about the center line and relate the array factor to the Chebyshev polynomial by using the transformation: $x = a\cos\psi + b$ and a polynomial of order $(N-1)/2$ for N elements. The visible region ranges from x_0 [Eq. (4-51)] at the beam peak to $x = -1$. Each root of the polynomial is used twice: once on the positive portion of the unit circle and once on the negative portion.

The beam peak occurs at $\psi = 0$ on the unit circle, which gives us $x_0 = a + b$. We set both the pattern start and finish on the unit circle to correspond to $x = -1$ of the Chebyshev poynomial; that is, $\psi_s = -kd + \delta$ and $\psi_f = kd + \delta$ transform into the point $x = -1$. Suppose $\delta = 0$ and $kd < \pi$ (less than λ/2 spacings, then

$$-1 = a\cos(kd) + b \tag{4-55.1}$$

Between $x_0 = a + b$ and Eq. (4-55.1), we solve for a and b.

$$a = \frac{1 + x_0}{1 - \cos kd} \quad \text{and} \quad b\,\frac{-(1 + x_0\cos kd)}{1 - \cos kd} \tag{4-55.2}$$

Example Design a Chebyshev array with nine elements at 0.375λ spacings and 25-dB sidelobes. We use the order $(N-1)/2 = 4$ for the Chebyshev polynomial. Equation (4-51) gives us $x_0 = \cosh[(\cosh^{-1}10^{25/20})/4] = 1.426$. We substitute the x_0 and $kd = 360°(0.375) = 135°$ into Eq. (4-55.2) to find the constants of the transformation: $a = 1.4209$ and $b = 0.0047$. We apply Eq. (4-52) to find the roots of the Chebyshev polynomial and the transformation $x = a\cos\psi + b$ to find the angles of the zeros:

$$\psi_p = \pm\cos^{-1}[(x_p - b)/a] \tag{4-55.3}$$

We substitute a, b, and the values of x_p found from Eq. (4-52) into Eq. (4-55.3) and obtain the following table, in which we obtain different zero locations ψ_p for both positive and negative solutions of Eq. (4-52) (x_p).

P	x_p	ψ_p, °	ψ_p, °
1	±0.9239	±49.69	±130.81
2	±0.3827	±74.57	±105.82

Given the zeros, we multiply out the polynomial to find the array coefficients. Figure 4-13.1 shows the unit circle representation and pattern of this design.

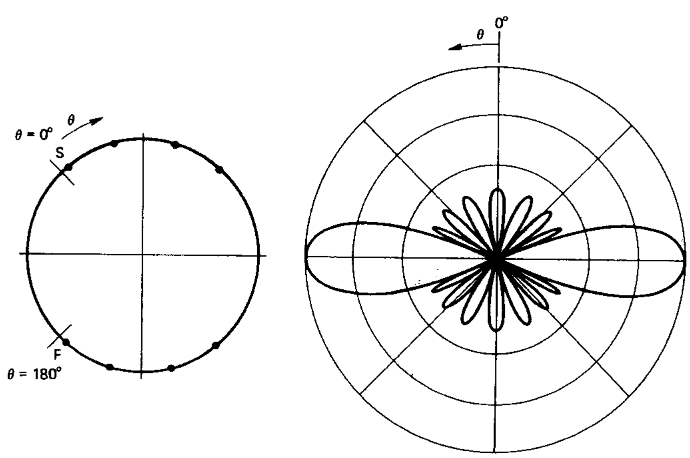

Figure 4-13.1 Nine-element Chebyshev array with 0.325λ spacings and 25-dB Sidelobes

Drane [33] gives an approximate formula for the directivity of these arrays with spacings less than λ/2.

$$directivity = \frac{2R^2}{1 + (\lambda/L)R^2\sqrt{(\ln 2R)/\pi}\,\sin(kd/2)} \tag{4-55.4}$$

Equation (4-55.4) gives good estimates for 10 or more elements. An estimate for the beamwidth of a scanned array is given by

$$BW = 10.3° \frac{\lambda}{L} \sqrt{S + 4.52} \csc \theta_0 \qquad (4\text{-}55.5)$$

where L is the array length (Nd), S the sidelobe level (dB), and θ_0 the scan angle from the axis. Equation (4-55.4) is also an estimate of the directivity of arrays with spacings greater than λ/2, but less than the appearance of grating lobes, if we remove the sin (kd/2) factor in the deniminator.

If kd is greater than or equal to π, the pattern will extend to or further than once around the unit circle in the W-plane. We use the following transformation

$$a = \frac{1 + x_0}{2} \quad \text{and} \quad b = \frac{x_0 - 1}{2} \qquad (4\text{-}55.6)$$

When we use this transformation, we obtain the same zeros ψ_p as found in section (4-10).

Suppose $\delta \neq 0$; then the beam will be scanned. We must adjust the zeros in the W-plane to still give an equal-sidelobe response. We consider two cases. In the first case the point W = -1 is excluded from the visible region. We modify Eq (4-55.2) to include δ in the transformation.

$$a = \frac{1 + x_0}{1 - \cos(kd - \delta)} \qquad b = \frac{-[1 + x_0 \cos(kd - \delta)]}{1 - \cos(kd - \delta)} \qquad (4\text{-}55.7)$$

Example Design a nine-element array with 0.25λ spacings scanned to 45° with 25-dB sidelobes.
To scan to 45° the progressive phase shift between elements δ is $-kd$ cos θ_m = -63.64°.
We use Eq. (4-51) to find x_0 = 1.4256 for $m = (N - 1)/2 = 4$. Substitute δ, kd, and x_0 into Eq. (4-55.7) to find a = 1.2793 and b = 0.1463. The combination of Eq. (4-52) for the roots x_p and Eq. (4-55.3) gives us the phase angles of the zeros on the unit circle.

P	x_p	ψ_p, °	ψ_p, °
1	±0.9239	±52.57	±146.77
2	±0.3827	±79.35	±114.42

In the second case the point W = -1 is in the visible region. We use Eq. (4-55.6) for a and b of the transformation. This method gives an equal-sidelobe design for an end-fire array, but it not optimum.

4-10.2 End-Fire Design

We let the beam maximum correspond to $-x_0$ [Eq. (4-51)], and make the progressive phase shift between elements δ greater than that necessary for end fire.

$$-x_0 = a \cos(kd - \delta) + b \qquad (4\text{-}55.8)$$

The pattern at ψ = 0 is at a maximum of the ripple. When x = 1,

$$a + b = 1 \qquad (4\text{-}55.9)$$

The other end of the visible region corresponds to x = -1:

$$-1 = a \cos(kd - \delta) + b \qquad (4\text{-}55.10)$$

We find the unknowns: a, b, and δ from Eqs. (4-55.8), (4-55.9), and (4-55.10).

$$a = \frac{x_0 + 3 + 2\cos(kd)\sqrt{2x_0 + 2}}{2\sin^2 kd}$$

$$b = 1 - a \qquad \delta = \sin^{-1} \frac{x_0 - 1}{2a \sin kd} \qquad (4\text{-}55.11)$$

Example Design a nine-element end-fire array with 0.25λ spacings and 25-dB sidelobes by using the expressions of Eq. (4-55.11).

From Eq. (4-51) x_0 = 1.4256 for m = 4. Substitute into Eq. (4-55.11) to find a = 2.2128, b = -1.2128, and δ = 5.52°. Equations (4-52) and (4-55.3) give us the angles of the zeros on the unit circle. The ψ_p values are

$$\psi_p = \pm 15.07, \pm 43.86, 67.97, \text{ and } \pm 82.50$$

Figure 4-13.2 is a plot of the W-space representation and array product. All the zeros have been moved into the visible region and produce a design with a large lobe in the invisible regions representing a large amount of stored energy in the aperture. We find array coefficients with alternating phases and a nearly 180° phase difference. This superdirective array will be narrrowband and have low efficiency.

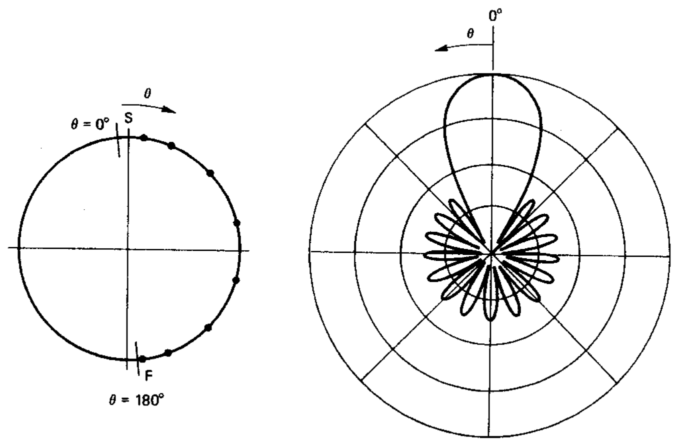

Figure 4-13.2 Nine-element Chebyshev array with λ/4 spacings and 25-dB Sidelobes

31. R. H. DuHamel, Optimum patterns for end-fire arrays, *Proceedings of IRE,* vol. 41, no. 5, May 1953, pp. 652-659.
32. W. L. Weeks, *Antenna Engineering*, McGraw-Hill, New York, 1968.
33. C. J. Drane, Useful approximations for the directivity and beamwidth of large scanning Dolph-Chebyschev arrays, *Proceedings of IRE*, vol. 56, no. 11, November 1968, pp. 1779-1787.